Legendrian links of topological unknots

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Abstract

We use an estimate on the Thurston–Bennequin invariant of a Legendrian link in terms of its Kauffman–polynomial to show that links of topological unknots, e.g. the Borromean rings or the Whithead link, may not be represented by Legendrian links of Legendrian unknots.

In [2] Eliashberg classified all Legendre knots representing the unknot in terms of their Thurston–Bennequin number, tb, and their rotation, r. Bennequin's inequality in these cases reads as

$$tb + |r| \leq -1.$$

Thus the Legendre knot given by the wavefront of the 'eye' which has $tb = -1, r = 0$ is referred to as the *trivial Legendrian knot*.

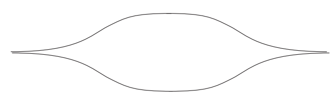

Figure 1: A front projection of the Legendrian unknot

Back then there was no other obvious obstruction for Legendrian *links* consisting of (topological) unknots and Eliashberg asked the question: "Given a link of topological unknots, can it be realized as a link of [. . .] Legendrian unknots?"

The answer is negative in general and the new obstructions are given by a sharper inequality on the Thurston–Bennequin number governed by the Kauffman polynomial $K(x,t)$ which was found by Lee Rudolph in [4] (for further reference see e.g. [1] and [3]). It simply states that the Thurston–Bennequin number is not bigger than the the minimal degree in the variable x of the Kauffman polynomial:

$$tb \leq -\operatorname{max-deg}_x K.$$

The contribution of the author is to apply this to links of topological unknots. We had to be careful because the two groups of authors [1, 3] used different Kauffman polynomials and thus obtain slightly different inequalities: here we work with the Dubrovnik–version Chmutov and Goryunov used. Let us first recall the definition of the Thurston–Bennequin number:

Definition 1 *Let L be an oriented Legendrian link given by a wave front projection. Then the* Thurston–Bennequin number *of L, $tb(L)$, is the number of sideward crossings minus the number of up– or downward crossings minus half the number of cusps.*

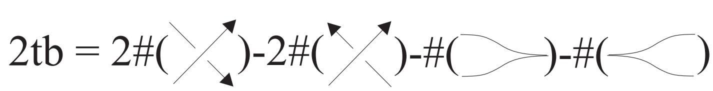

Figure 2: Combinatorial definition of the Thurston–Bennequin number in terms of the front projection

From that the following observation is immediate

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Lemma 2 *Let $L = \coprod_i L_i$ be a Legendrian link with pairwise unlinked components L_i. Then the Thurston–Bennequin number of that link is simply given by the sum of those of the components*

$$tb(L) = \sum_i tb(L_i).$$

In particular, it does not depend on the orientation of the components.

To investigate Eliashberg's question we took the most simplest examples we knew: The Borromean rings B and the Whitehead link W. The Kauffman polynomials are given by

$$\begin{aligned} K_W(x,y) = {} & yx^5 - 2x^4 - (2y^3+6y)x^3 + (-y^4 - y^2 + 6 + y^{-2})x^2 + (3y^3 + 9y + 2y^{-1})x \\ & + (y^4 + y^2 - 5 - 2y^{-2}) - (y^3 + 4y - 2y^{-1})x^{-1} + (2 + y^{-2})x^{-2} \end{aligned}$$

and

$$\begin{aligned} K_B(x,t) = {} & y^2x^4 + (-4y + y^{-3})x^3 + (-3y^4 - 10y^2 + 3y^{-2})x^2 \\ & + (-2y^5 - 2y^3 + 14y + 3y^{-1} - 3y^{-3})x + (6y^4 + 18y^2 + 1 - 6y^{-2}) \\ & - (-2y^5 - 2y^3 + 14y + 3y^{-1} - 3y^{-3})x^{-1} + (-3y^4 - 10y^2 + 3y^{-2})x^{-2} \\ & - (-4y + y^{-3})x^{-3} + y^2x^4. \end{aligned}$$

From that we easily deduce the main result of that note

Proposition 3 *(1) For any Legendrian representaion of the Borromean rings we have*

$$tb \leq -4.$$

(2) For any Legendrian representation of the Whitehaed link we have

$$tb \leq -5.$$

Thus for both not all components may be Legendrian unknots.

Remark 4 *(1) In the cases of the Borromean rings and the Whitehead link the inequality is sharp as the following pictures show.*
(2) In the case of the Whitehead link the Kauffman polynomial gives no further obstruction for its mirror. Indeed it can be represented as a Legendrian link of Legendrian unknots.

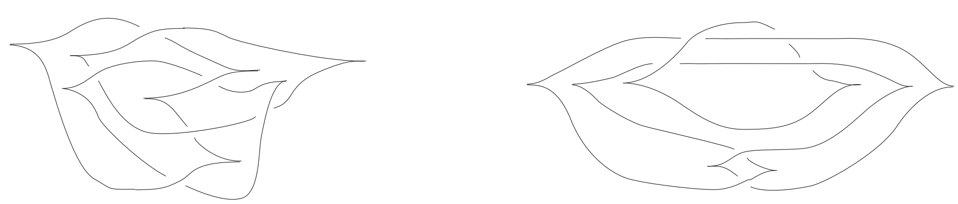

Figure 3: Legendrian Borromean rings with $tb = -4$ and Legendrian Whitehead mirror consisting of Legendrian unknots

On the other hand it is possible to give two different Legendrian Whithead links with $tb = -5$. One consists of components with Thurston–Bennequin numbers -4 and -1 the other with -3 and -2.

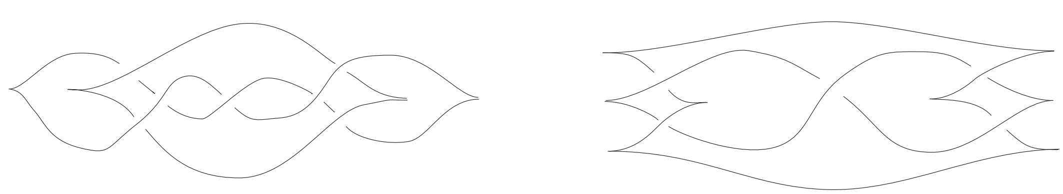

Figure 4: Two Legendrian Whitehead links with $tb = -5$

Questions 5 *(1) Are there sharper bounds on the Thurston–Bennequin number than that given in [1, 3]?*
(2) Are there sharper bounds if one imposes additional conditions? E.g. consider Brunnian links, i.e. links of unknots which fall appart if one removes one component.
(3) Is it at least true that a link of the type described in (2) may be represented as a Legendrian link consisting of Legendrian unknots iff the maximal degree in x of its Kauffman polynomial is equal to the number of components of the link (note that this degree is never less than the number of components)?
(4) Are there further restrictions for the distribution of the Thurston–Bennequin number on the components of a Legendrian link?

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References

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